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Dynamics in Imperfectly-Isochronous FFAG Accelerators

J.S. Berg
BNL, Upton, NY

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CENTER FOR ACCELERATOR PHYSICS

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J.S. Berg, Brookhaven National Laboratory, Building 901A, PO Box 5000, Upton, NY

Abstract

Using FFAGs for the arcs of recirculating accelerators has the potential to achieve significant cost savings over a multiple-arc design. However, no FFAG arc will have the same path length over its entire energy range. This leads to problems with synchronizing high-frequency RF with the beam on each pass. It has been demonstrated [1] that in fact a reference particle can be accelerated in such a system for an arbitrary number of turns, although the amount of linac required for a given energy gain never falls below a certain nonzero value for a larger number of turns. Here we examine that system in more generality, and begin to address longitudinal phase space acceptance.

1 LATTICE DESCRIPTION

For the purposes of this paper, a recirculating accelerator consists of an alternating sequence of identical linacs and arcs. The arcs are identical in that each has the same path length as a function of energy. The linacs are identical in that they all have the same voltage and the same phase. By "the same phase," I first mean that the phase of the RF does not change from one turn to the next. Second, if there are M linacs in the recirculating accelerator, the phase of one linac differs from that of the previous linac by $2\pi k/M$ for some fixed integer k.

There are two extremes in this design: one is the racetrack design, where there are two long parallel linacs connected by arcs; the opposite extreme is a distributed RF system, where one has a sequence of short arcs with a single RF cavity between them. The racetrack design allows one to attempt to suppress dispersion in the linacs, eliminating longitudinal-transverse coupling. It is generally difficult to suppress dispersion over a large energy range, and in addition, the dispersion suppression may reduce the dynamic aperture of the system. The distributed RF system allows longitudinal-transverse coupling, but maintains a high degree of symmetry, in principle giving a good dynamic aperture. The longitudinal-transverse coupling may not be so important, however, since we are on-crest, and the energy gain does not vary so strongly with time-of-flight (that variation is what causes the longitudinal-transverse coupling).

The path length in an FFAG arc is often well approximated as a quadratic function of energy (see Fig. 1 for an example). It is desirable to minimize the total variation in the path length over the desired energy range, and so one generally adjusts the lattice design to place the minimum of the parabola in the center of the energy range of the arc.
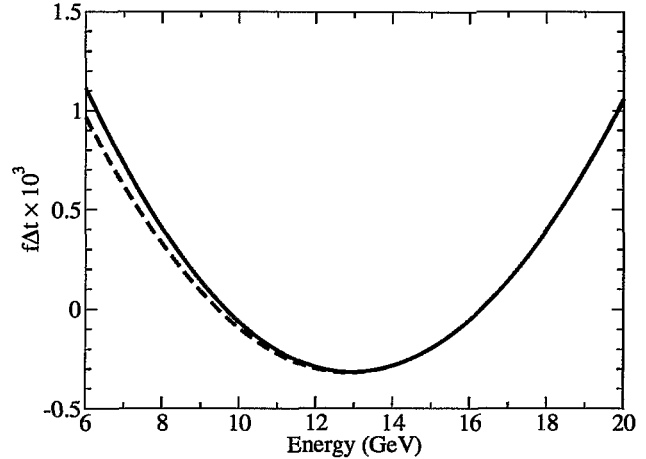
Figure 1: Path length as a function of energy for a single 6.5 m FFAG cell [2], calculated using COSY INFINITY [3]. Path length is given as a fraction of a 200 MHz RF period. The solid line is the exact path length variation, the dashed line is a quadratic approximation.

Thus, the path length as a function of energy takes the form

$$\Delta T \left(\frac{2E - E_{max} - E_{min}}{E_{max} - E_{min}} \right)^2 - T_0. \qquad (1)$$

ΔT should vary very little as T_0 is adjusted over a small range. Adjusting T_0 changes the energies for which the particle will see the same phase betweeen subsequent linacs. To minimize the relative arc lengthening required to adjust T_0, one can also adjust k described above (the relative phase between linacs).

Time-of-flight variation with energy is ignored in the linacs. Considering the relatively large energies that these recirculating accelerators are designed for, it is a very good approximation to distribute any path length variation with energy in the linacs into the adjacent arcs.

2 EQUATIONS OF MOTION

The equations giving the energy and time-of-flight at the entrance and exit of the linacs are

$$E_{n+1} = E_n + Vc(\omega\tau_n) \qquad (2)$$

$$\tau_{n+1} = \tau_n + \Delta T \left[u(E_{n+1}/\Delta E) - u_0 \right]. \qquad (3)$$

E_n is the energy after the nth pass through a linac, τ_{n-1} is the time-of-flight relative to the crest in the nth linac pass, ω is the angular RF frequency, and ΔE is the desired energy gain. $c(x)$ is a 2π periodic function whose maximum absolute value is 1 and whose integral over one period is 0, representing the amplitude of the voltage as a

function of phase. V is the maximum energy gained in the linac. $\Delta T[u(p) - u_0]$ is the time of flight in an arc as a function of energy (it is convenient to define $u(p)$ so that the difference between its maximum and minimum is 1). For the path length variation (1), $u(p) = 4(p - 1/2)^2$ and $u_0 \Delta T = T_0$. For a single RF frequency, $c(x) = \cos(x)$. Changing coordinates to $x_n = \omega \tau_n$ and $p_n = E_n/\Delta E$, (2-3) become

$$p_{n+1} = p_n + vc(x_n) \tag{4}$$

$$x_{n+1} = x_n + \Delta\phi[u(p_{n+1}) - u_0] \tag{5}$$

where $v = V/\Delta E$ and $\Delta\phi = \omega\Delta T$.

Say we want to accelerate from $E_{\max}$ to $E_{\min}$ in N turns. Then $p_0 = 0$ and $p_N = 1$. The problem that we wish to solve is given these endpoint conditions, minimize v by varying x_0, the phase at which you enter the first linac, and u_0. This solution will depend only on $\Delta\phi$ and N.

2.1 Continuous Approximation

Make a continuous approximation to Eqs. (4-5) [1]:

$$\frac{dp}{d\xi} = wc(x) \qquad \frac{dx}{d\xi} = u(p) - u_0, \tag{6}$$

where $w = V/(\Delta E \omega \Delta T)$ and $\xi = n\omega\Delta T$. The discrete problem is re-formulated to be that w is minimized subject to the constraints that $p(0) = 0$ and $p(N\omega\Delta T) = 1$ by varying u_0 and $x(0)$. This solution depends only on $N\omega\Delta T$. Thus, if the continuous approximation approximates the discrete system well, then V will be $\Delta E \omega \Delta T$ times a quantity depending only on $N\omega\Delta T$.

One can eliminate ξ from (6) and integrate to get

$$w[s(x) - s(x_0)] = U(p) - u_0 p, \tag{7}$$

where

$$s(x) = \int_0^x c(z)\, dz \qquad U(p) = \int_0^p u(z)\, dz. \tag{8}$$

From this we can show that

$$w \geqslant \frac{\sup\limits_{p \in [0,1]} [U(p) - u_0 p] - \inf\limits_{p \in [0,1]} [U(p) - u_0 p]}{\sup\limits_{x}[s(x)] - \inf\limits_{x}[s(x)]}. \tag{9}$$

Minimizing the right hand side over u_0 demonstrates that there is a nonzero lower bound on w. For example, for the path length variation (1), the numerator of (9) is 1/12, and is reached when $u_0 = 1/4$. When $c(x) = \cos(x)$, the denominator is 2, and thus w is at least 1/24. Setting w to the value computed on the right-hand side of (9) (even without minimizing with respect to u_0) and x_0 such that $s(x_0)$ is the minimum value of s ($-\pi/2$ when $c(x) = \cos x$), then if $U(p) - u_0 p$ reaches an extremum at an interior value of $p = p_1$, then $x = x_1$ and $p = p_1$ are a fixed point of (6), where $x = x_1$ solves (7) when $p = p_1$ (u and c are

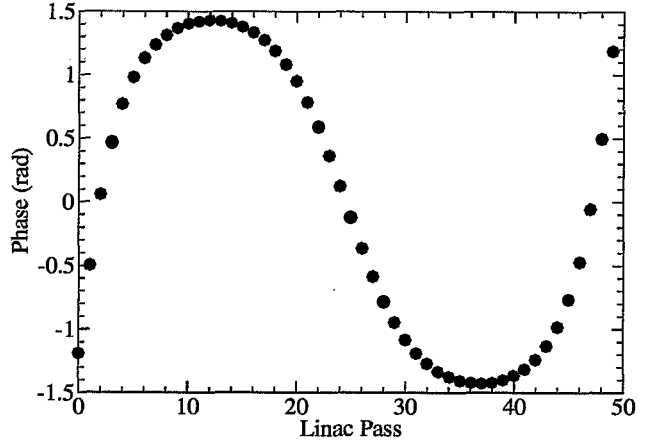
Figure 2: Phase as a function of linac pass when $\Delta\phi = 1$ and $N = 50$.

assumed to be differentiable). This fixed point must be unstable. Thus, it takes an infinite amount of time to reach that fixed point. Hence, the lower bound on w is the limit of the solution for w as $N\omega\Delta T \to \infty$. As we approach that limit, for w minimized over x_0, w approaches the right hand side of (9) from above, and x_0 approaches a value which minimizes $s(x)$.

This helps us understand the behavior in the discrete case: for large N, V will not go to zero but will approach a nonzero value which is proportional to $\Delta E \omega \Delta T$. The value of $V/(\Delta E \omega \Delta T)$ is roughly only a function of $N\omega\Delta T$. For large N, the bunch will spend many turns at a point where $c(x)$ is nearly zero, gaining very little energy. All these statements are weakly dependent on what $\omega\Delta T$ is, since the continuous approximation is not exact.

In the case of the distributed RF system, $\Delta\phi$ is small and N is very large, and the discrete equations are a very good approximation to the continuous ones. For a racetrack system, however, it is far from clear that the approximation is good. Since $\omega\Delta T$ is relatively large, the change of x in one step can be large, making it questionable whether the continuous approximation is really very good. However, we will subsequently see that for a large number of turns, a large fraction of the steps occur at points where the change in x_n is small, and x_n is large and therefore the change in p_n is also small. The continuous approximation thus turns out to give the correct results for large numbers of turns both qualitatively and nearly quantitatively as well.

3 EXAMPLE

We now find the minimum-v solution of Eqs. (4-5) for $\Delta\phi = 1$. This is a relatively large phase swing: remember that the phase errors accumulate, and so after only 4 steps with this phase error, one would certainly be decelerating. This example is more appropriate for a racetrack configuration: the phase swing per arc would be orders of magnitude smaller for a distributed RF system (but correspondingly more linac passes would be required).
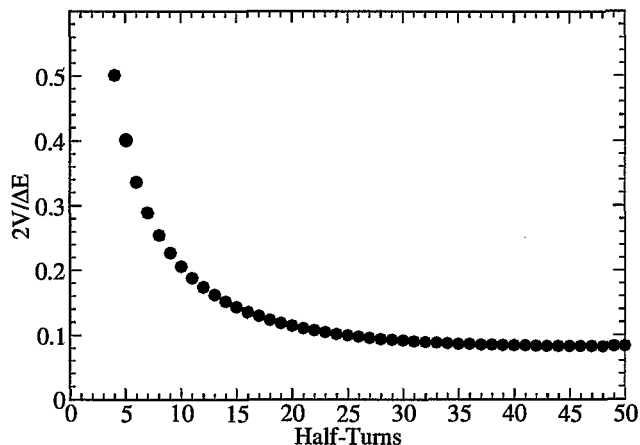
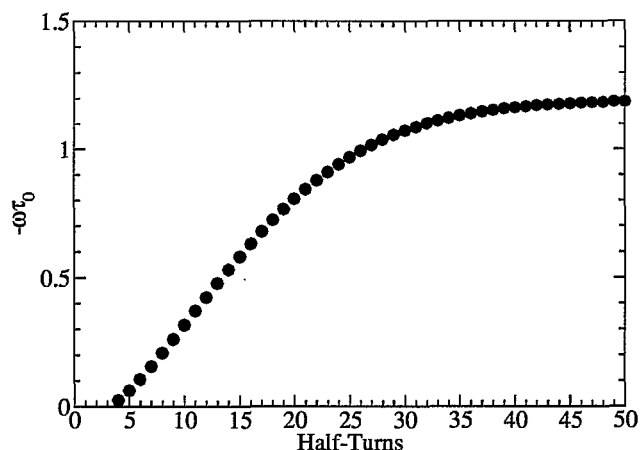Figure 3: Linac voltage versus the number of linac passes.

Figure 4: Initial phase versus the number of linac passes.

Figure 5: Initial phase space that is accepted in a 8 linac pass system. Each color represents a band of ± 0.01 in p_N.

Figure 6: Initial phase space that is accepted in a 24 linac pass system.

If one makes 50 linac passes for this system and performs the aforementioned optimization, the phase as a function of turn number is shown in Fig. 2. Note that the reference particle crosses the crest three times; this is related to the parabolic shape of the path length. The particle spends most of its time at the two turning points in phase (turns 8-18 and 32-42); these are the points where the path length error is near zero. Due to the large phase at this point, the particle is not gaining very much energy, and so remains at the point where the path length is near zero for a long time. This is what allows the particle can spend an arbitrary number of turns in this system.

Figure 3 shows the voltage as a function of the number of linac passes, and Fig. 4 shows x_0. For large numbers of turns, these (including ϕ_0, not shown) do appear to be approching the large N limits given above. While it is not clear that $x_0 \to -\pi/2$ in Fig. 4, from Fig. 2, one can see that the maximum phase swing is slightly larger than $-x_0$. The fact that $\Delta\phi$ is large causes this difference from the continuous approximation. It is the regime near the turning points in phase that approaches the continuous approximation. It turns out that the voltage limit as $N \to \infty$ is very slightly less than what is found in the continuous approxi-

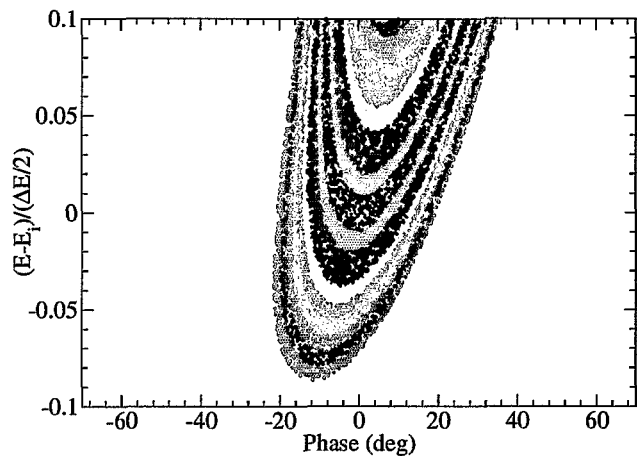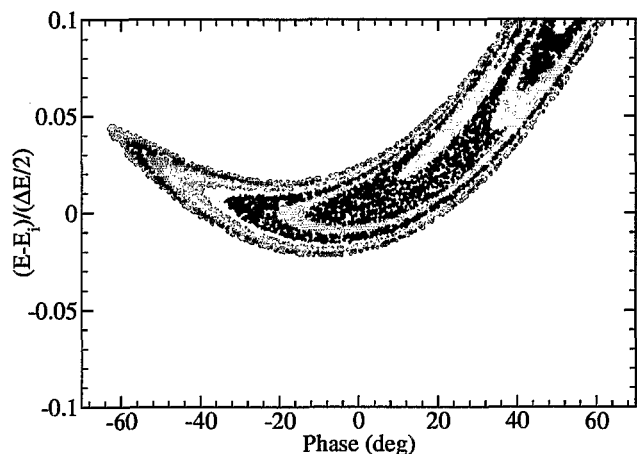mation; the difference is also due to the the the finite $\Delta\phi$.

Finally, Figs. 5 and 6 demonstrate how the acceptance varies with the number of turns. For fewer turns, a large region of phase space is accepted; what one sees at 8 linac passes is very close to what one expects from on-crest acceleration. For a large number of linac passes (24 here), most everything that is accepted ends up within a small band of the reference momentum at the final energy. In fact, while it appears that there is a smaller total acceptance for more turns, the phase space area ending up within a small energy band at the end is mmuch larger for more turns. The analysis of acceptance is still very preliminary, and must be studied more thoroughly.

4 REFERENCES

[1] J. Scott Berg. Proceedings of Snowmass 2001.

[2] Carol Johnstone. Private communication.

[3] K. Makino and M. Berz. COSY INFINITY version 8. *Nuclear Instruments and Methods*, A427, pp. 338–343 (1999).